Supplementary Problems for

Combinatorics Problems and Solutions Second Edition

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Supplementary Combinatorics Problems

Problem 1. Show that the number of ways in which two people can divide $2n$ things of one kind, and $2n$ of another kind, and $2n$ of a third kind, so that each person gets $3n$ things is $3n^2 + 3n + 1$.

Answer 1. All 7 ways for $n = 1$ with A, B, C being the 3 kinds of things, is shown in table 1. The

	person 1 ABC	person 2 ABC
1	111	111
2	021	201
3	012	210
4	201	021
5	102	120
6	210	012
7	120	102

Table 1: All 7 ways for $n = 1$ in problem 1.

table provides the perspective to state the problem a different way: In how many ways can you distribute $3n$ identical balls into 3 distinct bins so that no bin contains more than $2n$ balls?

By way #2 of the twelve fold way, the number of ways to distribute $3n$ identical balls into 3 distinct bins is $\binom{3n+2}{2}$.

The number of ways to distribute the balls so that at least one bin has more than $2n$ balls is $3\binom{n+1}{2}$. To get this expression set aside $2n+1$ of the $3n$ balls and distribute the remaining $n-1$ balls into the 3 bins in $\binom{n+1}{2}$ ways. The remaining $2n+1$ balls can then be put into one of the bins in 3 ways.

The number of ways to distribute the balls so that no bin has more than $2n$ balls is then

$$\binom{3n+2}{2} - 3\binom{n+1}{2} = 3n^2 + 3n + 1$$

Problem 2. Give a combinatorial proof of the following identity

$$\sum_{k=0}^{n} \binom{n-k+m-1}{m-1} = \binom{n+m}{m}$$

Answer 2. $\binom{n+m}{m}$ is equal to the number of ways to distribute n indistinguishable objects into $m+1$ distinguishable bins. Pick one of the bins. Over the set of all distributions that bin will hold between 0 and n objects. If it holds k objects then the remaining $n-k$ objects will be distributed into the other m bins in

$\binom{n-k+m-1}{m-1}$ ways. Summing this over all the values of k we get the identity.

Problem 3. Given a collection of n identical red balls, n identical green balls and n identical blue balls, in how many ways can the $3n$ balls be distributed into 3 bins such that each bin contains exactly n balls?

Answer 3. We only need to count the number of ways to distribute n balls into each of two bins since the remaining n balls will then go into the remaining bin. This is equivalent to asking for the number of ways to distribute n identical balls into bins labeled R_1, G_1, B_1 and n identical balls into bins labeled R_2, G_2, B_2 such that $|R_1| + |R_2| \leq n$, $|G_1| + |G_2| \leq n$, $|B_1| + |B_2| \leq n$. The vertical bars around a bin label means the number of balls in that bin. Without these restrictions each of the distributions of n balls can be done in $\binom{n+2}{2}$ ways so the total number of ways to distribute the $2n$ balls without restrictions is $\binom{n+2}{2}\binom{n+2}{2}$. From this we have to subtract the number of distributions in which one of the conditions $|R_1| + |R_2| > n$, $|G_1| + |G_2| > n$, $|B_1| + |B_2| > n$ holds. Note that since there are $2n$ balls only one of the conditions can hold for a given distribution. Suppose for example that we have $|R_1| + |R_2| = n + k$ where $k = 1, , 2, \ldots, n$. The rest of the $n - k$ balls can be distributed into the remaining 4 bins in $\binom{n-k+3}{3}$ ways. Summing this over all values of k we get the total number of ways that we can have $|R_1| + |R_2| > n$.

Using the combinatorial identity proven in the previous problem we have

$$\sum_{k=1}^{n} \binom{n-k+3}{3} = \binom{n+3}{4}$$

To put this in the form of the identity change the summation index to $k' = k-1$. So that we have (dropping the prime on k)

$$\sum_{k=0}^{n-1} \binom{n-1-k+3}{3} = \binom{n-1+4}{4} = \binom{n+3}{4}$$

The total number of ways to distribute $3n$ balls into 3 bins such that each bin contains exactly n balls is then

$$a(n) = \binom{n+2}{2}\binom{n+2}{2} - 3\binom{n+3}{4}$$

where the 3 multiplying $\binom{n+3}{4}$ comes from the fact that we 3 conditions that have to be accounted for. We can also write the answer without the binomials as

$$a(n) = \frac{1}{8}(n+1)(n+2)(n^2+3n+4)$$

Table 2 shows the value of $a(n)$ for $n = 0, 1, \ldots, 8$ and table 3 shows the 21 ways to distribute the balls $\{rrggbb\}$ into 3 bins with 2 balls per bin.

Problem 4. In how many ways can 6 lilies, 7 roses and 10 tulips be arranged in a row so that each lily is

n	0	1	2	3	4	5	6	7	8
a(n)	1	6	21	55	120	231	406	666	1035

Table 2: $a(n)$ values.

Bin 1	Bin 2	Bin 3
rr	gg	bb
rr	gb	gb
rr	bb	gg
rg	rg	bb
rg	rb	gb
rg	gb	rb
rg	bb	rg
rb	rg	gb
rb	rb	gg
rb	gg	rb
rb	gb	rg
gg	rr	bb
gg	rb	rb
gg	bb	rr
gb	rr	gb
gb	rg	rb
gb	rb	rg
gb	gb	rr
bb	rr	gg
bb	rg	rg
bb	gg	rr

Table 3: The 21 ways to distribute the balls {rrggbb} into 3 bins with 2 balls per bin.

between a rose and a tulip, and there are no roses and tulips next to each other?

Answer 4. Let L represent a lily, and R and T represent a group of roses and tulips respectively. There are then 2 possible arrangements: $RLTLRLTLRLTLR$, $TLRLTLRLTLRLT$. Call these arrangements 1 and 2 respectively. In arrangement 1 there are 4 R groups and 3 T groups. The ways to divide 7 roses into 4 groups with at least one in each group is $\binom{6}{3}$. The ways to divide 10 tulips into 3 groups with at least one in each group is $\binom{9}{2}$. The total number of ways to create arrangement 1 is then $\binom{6}{3}\binom{9}{2}$. In arrangement 2 there are 3 R groups and 4 T groups. The ways to divide 7 roses into 3 groups with at least one in each group are $\binom{6}{2}$. The number of ways to divide 10 tulips into 4 groups with at least one in each group is $\binom{9}{3}$. The total number of ways to create arrangement 3 is $\binom{6}{2}\binom{9}{3}$. The total number of arrangements is then

$$\binom{6}{3}\binom{9}{2} + \binom{6}{2}\binom{9}{3} = 1980$$

Problem 5. Show that $\binom{n}{k}$ is a maximum when $k = \lfloor n/2 \rceil$, i.e. when k is equal to the nearest integer to $n/2$.

Answer 5. If $\binom{n}{k}$ were a continuous function we would take the derivative with respect to k set it equal to zero and solve for k. We can do basically the same thing here by calculating the central difference.

$$\frac{1}{2}\left[\binom{n}{k+1}-\binom{n}{k-1}\right]=$$
$$\frac{1}{2}\left[\frac{n!}{(k+1)!(n-k-1)!}-\frac{n!}{(k-1)!(n-k+1)!}\right]$$

Now factor out $\binom{n}{k}$ from the two terms and set the result equal to zero.

$$\frac{1}{2}\binom{n}{k}\left[\frac{n-k}{k+1}-\frac{k}{n-k+1}\right]=0$$

So we must have

$$\frac{n-k}{k+1}=\frac{k}{n-k+1}$$

Solving for k we get $k=n/2$.

Another way to solve this problem is to use Stirling's approximation $\ln n! \approx n \ln n - n$. Then we can find the maximum of $\ln \binom{n}{k}$ which is the same as the maximum of $\binom{n}{k}$.

$$\ln \binom{n}{k} = \ln n! - \ln k! - \ln (n-k)!$$

Using Stirling's approximation this becomes

$$\ln \binom{n}{k} \approx n \ln n - n - k \ln k + k - (n-k) \ln (n-k) + n - k$$

Now take the derivative with respect to k and set the result equal to zero.

$$\ln(n-k) - \ln k = \ln\left(\frac{n-k}{k}\right) = 0$$

Solving for k gives $k = n/2$.

Problem 6. Show that the product of binomial coefficients

$$\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_q}{k_q}$$

where $n_1 + n_2 + \cdots + n_q = n$ and $k_1 + k_2 + \cdots + k_q = k$, is a maximum when the k_i values are equal to

$$k_i = \frac{n_i k}{n}$$

Answer 6. The only reasonable way to solve this problem is to use Stirling's approximation for the log of a factorial, as we did in the previous problem. Taking the log of the binomial products we have

$$\sum_i \left[\ln n_i! - \ln k_i! - \ln(n_i - k_i)!\right]$$

Using Stirling's approximation this can be simplified to

$$H = \sum_i \left[n_i \ln n_i - k_i \ln k_i - (n_i - k_i)\ln(n_i - k_i)\right]$$

Now we want to maximize H subject to the constraint that

$$\sum_i k_i = k$$

We will do this using the method of Lagrange multipliers. First we construct the function

$$L = H + \lambda \left(\sum_i k_i - k \right)$$

Now we have the two equations

$$\frac{\partial L}{\partial \lambda} = \sum_i k_i - k = 0$$

$$\frac{\partial L}{\partial k_i} = \frac{\partial H}{\partial k_i} + \lambda = 0$$

These two equations can be solved for k_i. From the expression for H we have

$$\frac{\partial H}{\partial k_i} = \ln \left(\frac{n_i - k_i}{k_i} \right)$$

The previous equation then becomes

$$\ln \left(\frac{n_i - k_i}{k_i} \right) + \lambda = 0$$

which we can write as

$$\frac{n_i - k_i}{k_i} = e^{-\lambda}$$

Solving this for k_i we have

$$k_i = \frac{n_i}{1 + e^{-\lambda}}$$

Summing this equation over i we get

$$k = \frac{n}{1 + e^{-\lambda}}$$

So that

$$1 + e^{-\lambda} = \frac{n}{k}$$

Substituting this into the equation for k_i we get

$$k_i = \frac{n_i k}{n}$$

In quantum mechanics, particles (electrons, protons, atoms, etc.) that are bound by a potential energy function will have discrete energy levels. A particle in a box where the walls are infinite potential energy barriers is probably the simplest example. In the one dimensional case the particle is confined to a one dimensional region of some fixed length. The energy levels of the particle are limited to the values $E_n = an^2$ where a is a constant and $n = 1, 2, 3 \ldots$.

Another example is the energy levels of an electron in a hydrogen atom. The electron is limited to the energies $E_n = -13.6/n^2$ where $n = 1, 2, 3 \ldots$. A system where the energy levels are equally spaced is the quantum harmonic oscillator which corresponds to the classical system of a particle oscillating on the end of a spring. Here the energy levels have the form $E_n = an+b$ where a and b are constants and $n = 0, 1, 2, \ldots$.

So in general a particle in a quantum system will have one of a set of discrete energy levels E_n. At each energy level there will be a finite set of states the particle can be in. These states may correspond for example to allowed angular momenta for an electron bound to an atom. For the sake of the following discussion, you can picture the particles as being organized into boxes on a set of shelves. Each shelf is an energy level and the boxes on the shelf are the states. In general each shelf may have a different number of boxes.

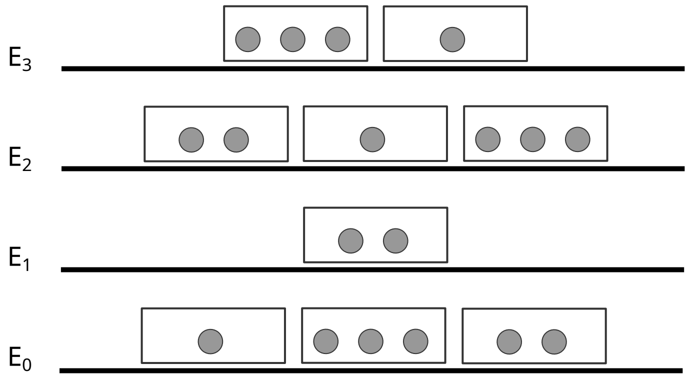

Figure 1: The distribution of particles in a quantum system is combinatorially equivalent to distributing balls into boxes on a shelf. The shelves correspond to energy levels and the boxes correspond to quantum states.

The properties of a system composed of a very large number of particles is determined by the way the particles distribute themselves among the energy levels and states. That distribution is determined by the total energy of the system and by the type of particles. In quantum mechanics there are two types of particles called bosons and fermions with very different rules for how they may occupy states.

Bosons have no restrictions on how many of them may occupy the same state simultaneously. Any number of them may bunch up together in the same state. Fermions on the other hand are more standoffish. Only one fermion may occupy a given state at a time. These properties of bosons and fermions determine the total number of ways they can be distributed among the states.

Problem 7. An energy level in a quantum system has n_j particles. The level has g_j states that can be occupied by the particles. If the particles are bosons how many ways can they be distributed among the states?

Answer 7. Bosons are identical, indistinguishable particles and any number of them may occupy a given state. So combinatorially this is equivalent to finding the number of ways that n_j identical balls can be placed into g_j distinguishable boxes. From the twelve fold way the number of distributions is

$$\binom{n_j + g_j - 1}{g_j - 1}$$

Problem 8. In the previous problem let the particles be fermions instead of bosons.

Answer 8. Fermions are also identical and indistinguishable particles but only one of them may occupy a given state at a time. For a distribution to be possible the number of fermions must be less than or equal to the number of states, $n_j \leq g_j$. The number of distributions is the number of ways to choose n_j out of the g_j states for occupation by particles. That number is

$$\binom{g_j}{n_j}$$

Problem 9. Suppose we have a system with a fixed number of bosons, N, and a fixed energy, U. The system has integer valued energy levels, $\epsilon_j = j$ for $j = 0, 1, 2, \ldots$ and each energy level has only one state. If $N = 6$ and $U = 5$ how many ways can the particles be distributed among the energy levels?

Answer 9. We can answer this question by first looking at the number of ways the energy can be partitioned. The partitions of 5 are

1. $\{1, 1, 1, 1, 1\}$
2. $\{1, 1, 1, 2\}$
3. $\{1, 1, 3\}$
4. $\{1, 2, 2\}$
5. $\{1, 4\}$
6. $\{2, 3\}$

7. {5}

There are 7 partitions. Partition 1 corresponds to having 5 particles with energy equal to 1 and 1 particle with energy equal to 0. Partition 2 corresponds to 3 particles with energy equal to 1, 1 particle with energy equal to 2, and 2 particles with energy equal to 0. Partition 3 corresponds to 2 particles with energy equal to 1, 1 particle with energy equal to 3, and 3 particles with energy equal to 0. And so on for the other partitions.

The particles are bosons so they are indistinguishable and any number of them can be in the same state. There is therefore only one way to arrange the particles for each partition. The answer is then simply equal to the number of partitions which in this case is 7.

Problem 10. Repeat the previous problem for the case of distinguishable classical particles.

Answer 10. We still have 7 partitions of the energy as in the previous problem but now there is more than one way to arrange the particles for each partition. For example in partition 1 there are 6 ways one of the particles can have energy equal to 0. In general for a given partition if there are N_j particles with energy j then the number of ways to arrange the particles is

$$\frac{N!}{N_0!N_1!\cdots}$$

So the number of ways the particles can be arranged in each of the partitions is

1. $\frac{6!}{1!5!} = 6$
2. $\frac{6!}{2!3!1!} = 60$
3. $\frac{6!}{3!2!1!} = 60$
4. $\frac{6!}{3!1!2!} = 60$
5. $\frac{6!}{4!1!1!} = 30$
6. $\frac{6!}{4!1!1!} = 30$
7. $\frac{6!}{5!1!} = 6$

Summing the number of arrangements for each partition we get 252 for the total number of arrangements.

Problem 11. Repeat the previous problem for the case of fermions.

Answer 11. The number of partitions of the energy does not change. Since the particles are fermions and there is only one state for each energy level only one particle can occupy a given level. If we have 6 particles then none of the partitions meet this requirement so the number of arrangements is 0. Such a system cannot be populated with 6 fermions.

Problem 12. If in the previous problem we have 2 fermions instead of 6 how many ways can they be arranged?

Answer 12. With only 2 fermions partitions 5, 6 and 7 can be occupied. Each of the partitions can be occupied in one way so the total number of arrangements is 3.